

# One-dimensional model of cosmological perturbations: direct integration in the Fourier space

V.M. Sliusar<sup>1</sup>, V.I. Zhdanov<sup>2</sup>

<sup>1,2</sup>*Astronomical Observatory, National Taras Shevchenko University of Kyiv, Observatorna str.,3, Kiev 04053, Ukraine*

<sup>1</sup> vitaliy.slyusar@gmail.com

<sup>2</sup> ValeryZhdanov@gmail.com

## Abstract

We propose a method of calculation of the power spectrum of cosmological perturbations by means of a direct numerical integration of hydrodynamic equations in the Fourier space for a random ensemble of initial conditions with subsequent averaging procedure. This method can be an alternative to the cosmological N-body simulations. We test realizability of this method in case of one-dimensional motion of gravitating matter pressureless shells. In order to test the numerical simulations, we found an analytical solution which describes one-dimensional collapse of plane shells. The results are used to study a nonlinear interaction of different Fourier modes.

*Key words:* large scale structure, cosmological perturbations, hydrodynamics.

## 1 Introduction

Theoretical investigation of the cosmological inhomogeneity growth presents one of the most serious challenges to computational astrophysics. A number of problems arises on non-linear scales when the density contrast cannot be assumed small. These include simulations of the galaxy formation process, working out predictions concerning the galactic environment (number of

dwarf satellite galaxies) and structure of their central regions (the cusp-core problem) either with cold dark matter (DM) or within warm DM models [1, 2, 3, 4, 5, 6]. Interesting possibility to obtain bounds on masses of DM particles stems from observations of Ly- $\alpha$  forest (see, e.g., [7]) and references therein), which requires accurate calculations of the power spectrum of cosmological inhomogeneity on kiloparsec scales.

Most developed computational techniques to study the cosmological structure formation involve N-body simulations combined with the smoothed particle hydrodynamics [8, 9, 10]. Currently performed simulations involve up to  $10^9$  particles (see, e.g., [9, 6]). These methods should be tested in independent simulations.

On the other hand, some analytical and semi-analytical schemes were proposed to study the matter power spectra on small scales [11, 12, 13]. These typically involve perturbative schemes after transition to the Fourier-transformed hydrodynamical variables. There are techniques dealing with correlation functions in the Fourier space [14, 15]. These authors use some additional *a priori* suggestions in order to close the infinite chain of correlation functions. The validity of these suggestions is not evident. A comparison of different approaches can be found in [16].

In this paper we propose an alternative method, which uses a direct integration of hydrodynamical equations in the Fourier space. In order to estimate workability of the method, as a first step we consider a one-dimensional problem of hydrodynamical evolution for a pressureless gravitating matter, i.e. one-dimensional density shells. The integration is performed for each realization from a random ensemble of initial data with subsequent averaging procedure. In order to test the numerical simulations, we found an analytical solution which describes one-dimensional collapse of the plane shells (Section 2). This solution is used in case of periodic initial data corresponding to a symmetric motion. In section 3 we write down the equations for the Fourier coefficients and present some results for the power spectrum obtained after the statistical averaging.

## 2 Implicit analytical solution

In order to test numerical simulations, it is useful to have an exact solution. In this section we obtain such a solution of one-dimensional problem in the Lagrange variables. The 1-dimensional version of hydrodynamical equations

(continuity, Euler and Poisson equations) in case of a pressureless gravitating fluid is:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho V) = 0, \quad (1)$$

$$\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} = -\frac{\partial \Phi}{\partial x}, \quad (2)$$

$$\frac{\partial^2 \Phi}{\partial x^2} = 4\pi G \rho, \quad (3)$$

$V$  is the velocity,  $\rho$  is the mass density,  $\Phi$  is the gravitational potential,  $G$  is the gravitational constant. In order to pass to the Lagrangian description we introduce the stream lines  $X(\xi, t)$ :

$$\frac{dX}{dt} = V(X, t), \quad X(\xi, 0) \equiv \xi \quad . \quad (4)$$

Equation (2) yields

$$\frac{d^2 X}{dt^2} = E(X, t), \quad E(x) = -\frac{\partial \Phi}{\partial x} \quad (5)$$

Let for  $t = 0$ :  $\rho(x, 0) = \rho_0(x)$ ,  $V(x, 0) = V_0(x)$ . Using (1), (2) we see that  $E(X(\xi_1, t), t) - E(X(\xi_2, t), t) = \text{const}$  is constant along the stream lines. This is a consequence of mass conservation in plane layer between  $X(\xi_1, t)$  and  $X(\xi_2, t)$ . Then we obtain because of (5)

$$X(\xi_1, t) - X(\xi_2, t) = \xi_1 - \xi_2 + t [V_0(\xi_1) - V_0(\xi_2)] - 2\pi G t^2 \int_{\xi_2}^{\xi_1} dx \rho_0(x). \quad (6)$$

Formula (6) allows to get a general solution of the one-dimensional problem in Lagrangian coordinates. Further for simplicity we deal with a symmetric motion (with respect to the coordinate origin  $\xi = 0$ ) so that  $X(0, t) \equiv 0$ ,  $V_0(0) = 0$ . In this case

$$X(\xi, t) = \xi + t V_0(\xi) - 2\pi G t^2 \int_0^{\xi} dx \rho_0(x) \quad (7)$$

The solution is valid during a limited time until

$$\frac{\partial X}{\partial \xi} = 1 + t V_0'(\xi) - 2\pi G t^2 \rho_0(\xi) \neq 0 \quad (8)$$

Transition to Euler variables is

$$\rho(x, t) = \rho_0(\xi) \left( \frac{\partial X}{\partial \xi} \right)^{-1}, \quad V(x, t) = V_0(\xi) - 4\pi G t \int_0^\xi dx \rho_0(x) \quad (9)$$

where  $\xi = \xi(x, t)$  is defined implicitly as the solution of the equation:

$$x = X(\xi, t) \quad (10)$$

In case of homogeneous initial conditions  $\rho(x, 0) = \rho_{h0}(x) = \text{const}$ ,  $V(x, 0) \equiv V_{h0}(x) = H_0 x$ ,  $H_0$  is the "Hubble constant", we get:

$$x = \xi (1 + H_0 t - 2\pi G \rho_{h0} t^2), \quad \rho_h(x, t) = \frac{\rho_{h0}}{1 + H_0 t - 2\pi G \rho_{h0} t^2},$$

$$V_h(x, t) = x \frac{H_0 - 4\pi G t \rho_{h0}}{1 + H_0 t - 2\pi G \rho_{h0} t^2} \quad (11)$$

or

$$\rho_h(x, t) = \frac{\rho_{h0}}{R(t)}, \quad V_h(x, t) = x \frac{\dot{R}(t)}{R(t)}, \quad R(t) = 1 + H_0 t - 2\pi G \rho_{h0} t^2. \quad (12)$$

To consider deviations from the homogeneous background, we impose periodic boundary conditions such that the initial conditions ( $t = 0$ ) are as follows:

$$V_0(\xi) = H_0 \left[ \xi + \sum_{n=1}^{\infty} \frac{a_n^0}{k_n} \sin(k_n \xi) \right], \quad \rho_0(\xi) = \rho_{h0} \left[ 1 + \sum_{n=1}^{\infty} b_n^0 \cos(k_n \xi) \right],$$

where  $L$  is the periodic "box" size,  $k_n = 2\pi n/L$ .

Substitution of the initial conditions into (7-9) yields

$$X(\xi, t) = R(t)\xi + tH_0 \sum_{n=1}^{\infty} \frac{a_n^0}{k_n} \sin(k_n \xi) - \frac{3}{4} (tH_0)^2 \sum_{n=1}^{\infty} \frac{b_n^0}{k_n} \sin(k_n \xi),$$

$$\rho(x, t) = \frac{\rho_{h0} \left[ 1 + \sum_{n=1}^{\infty} b_n^0 \cos(k_n \xi) \right]}{R(t) + tH_0 \sum_{n=1}^{\infty} \frac{a_n^0}{k_n} \cos(k_n \xi) - \frac{3}{4} (tH_0)^2 \sum_{n=1}^{\infty} b_n^0 \cos(k_n \xi)},$$

The solution shows a singular growth (collapse of one-dimensional gravitating layers) as the condition (8) is violated.

At the end of this Section we note that in spite of the exact form of the solution given by (7-9) we cannot avoid a numerical work when we pass to the Euler variables and then to the Fourier representation. So we refer to the method of this Section as "semi-analytical".

### 3 Numerical simulations

Furthermore for the homogenous background we denote  $\mathcal{H}(\tau) \equiv dR/dt$ ,  $\tau$  is the "conformal time":  $dt = R(\tau)d\tau$ , and  $y = x/R(\tau)$  is the comoving spatial coordinate. After some calculation on account of (12) we get

$$\tau = \frac{1}{2H_0} \ln \left[ \frac{1 + 3H_0 t}{3(2 - H_0 t)} \right], \quad \mathcal{H}(\tau) = \frac{dR}{dt} = \frac{1}{R} \frac{dR}{d\tau} = -2H_0 \tanh(H_0 \tau).$$

Hereafter  $\delta$  is the density contrast,  $\theta = \partial v / \partial y$ ,  $v$  is the peculiar velocity. Taking into account Poisson equation (3) we get:

$$\frac{\partial^2 \phi}{\partial y^2} = \alpha R(\tau) \delta, \quad \alpha = 4\pi G \rho_{h0}, \quad (13)$$

In terms of conformal  $\tau$  and comoving  $y$  the hydrodynamic equations can be written as:

$$\frac{\partial \delta}{\partial \tau} + \theta + \frac{\partial}{\partial y} (v \delta) = 0 \quad (14)$$

$$\frac{\partial \theta}{\partial \tau} + H \theta + \frac{\partial}{\partial y} (\theta v) = -\alpha R \delta, \quad (15)$$

We proceed to deal with the Fourier coefficients in the symmetric one-dimensional case.

$$\delta(x, \tau) = \sum_{n=-\infty}^{\infty} b_n(\tau) \exp(ik_n x), \quad k_n = 2\pi n/L, \quad (16)$$

$$\theta(x, \tau) = \sum_{n=-\infty}^{\infty} a_n(\tau) \exp(ik_n x), \quad v(x, \tau) = \sum_{n=-\infty}^{\infty} \frac{a_n(\tau)}{ik_n} \exp(ik_n x). \quad (17)$$

The reverse transformation is:

$$b_n(\tau) = L^{-1} \int_0^L dx e^{ik_n x} \delta(x, \tau), \quad a_n(\tau) = L^{-1} \int_0^L dx e^{ik_n x} \theta(x, \tau) \quad (18)$$

We assume  $a_0 = 0$ ,  $b_0 = 0$  at  $t = 0$  then it is easy to see from (14,15) that these equalities are fulfilled for all  $t > 0$ .

The equations for the Fourier coefficients take on the form:

$$\frac{da_n}{d\tau} + H(\tau)a_n + \alpha R(\tau)b_n + n \sum_{\substack{p=-\infty \\ p \neq 0}}^{\infty} \frac{a_p a_{n-p}}{p} = 0, \quad n = \pm 1, \pm 2, \dots \quad (19)$$

$$\frac{db_n}{d\tau} + a_n + n \sum_{\substack{p=-\infty \\ p \neq 0}}^{\infty} \frac{a_p b_{n-p}}{p} = 0, \quad n = \pm 1, \pm 2, \dots \quad (20)$$

The numerical solution of the equations (19), (20) was performed using the 4-th order Runge-Kutta method after a truncation of the infinite chain of coefficients  $a_n, b_n$ . Calculations were carried out by the specially written GPGPU code using OpenCL SDK by AMD. The time, required to calculate  $a_n$  and  $b_n$ , in case of 256 values of  $n$  (points over  $k$ ) for single realization of initial conditions, is 10 seconds. This is considerably faster than direct using of the semi-analytical solution of Section 2. It is important to note that it is easy to extend the corresponding algorithms to the 3-D case.

We calculated coefficients  $b_n$  of the density contrast as a function of  $t$  by means of analytical and numerical methods with the same initial conditions. On the Fig. 1 these coefficients are presented for  $t = 0.9$  and  $t = 1.7$  for both methods. For larger  $t$  we observe an infinite growth (for finite time) that corresponds to collapse of some plane gravitating shells due violation of condition (8). Correspondingly, the difference between two methods, that reflects the error of calculation, increases for greater  $t$  and greater  $n$ . For example, in order to look how the perturbation propagates from small wavenumbers to larger ones, we considered the following initial conditions:  $b_n(0) = 0$  where integer  $n$  varies  $-128$  to  $128$  except  $n = \pm 1$ ;  $b_{\pm 1}(0) = b_{\pm 1}^0/2 = 0.1$ ; all  $a_n(0) = 0$ . For  $t = 0.2$  or  $t = 0.5$  the difference between the values calculated by different methods is less than 1%, and for  $t = 1.7$  the difference changes from 1.5% to 6.7% as  $k_n = 2\pi n/L$  increases from 0.6 to 6. Larger  $k$ -interval is presented on Fig. 2. The next figure presents the power spectrum obtained

by averaging of the solutions for the ensemble of initial data with uniform distribution of  $b_{\pm 1}(0)$ ,  $\langle b_{\pm 1}^2(0) \rangle = 0.5$ . We observe the growth of dispersion, which is explained as follows: as  $t$  grows, some of realizations of the ensemble of solutions (with larger  $|b_{\pm 1}(0)|$ ) enter the region which is close to the singularity.

## 4 Conclusions

We present a new approach to investigation of the cosmological inhomogeneity by means of the direct integration of hydrodynamic equations in the Fourier space. At the moment we studied a one-dimensional hydrodynamical evolution of cold (pressureless) gravitating matter. The numerical integration has been fulfilled for a random ensemble of initial conditions with subsequent averaging procedure to get the power spectrum of the density contrast. We used the GPGPU instead of the classic CPU because the problem can be easily processed in parallel environment.

The numerical simulations have been tested using the analytic solution that describes the one-dimensional collapse of gravitating shells. The density contrast shows a propagation of perturbations from smaller wavenumbers to larger ones. The evolution in time ends with a singular growth of the density contrast. Correspondingly, we point out a significant growth of dispersion of the power spectrum in the non-linear region.

We consider our results as a first step to the simulations of cosmological inhomogeneity growth in the cold matter that could be an alternative to the cosmological N-body simulations. We expect that our approach will be especially effective in the weakly nonlinear regime. The next step will be an implementation of the three-dimensional case of the problem, which is technically similar to the one dimensional problem. The trial runs of our method allow us to think that it could be really used for power spectrum calculations in the 3-D case with realistic requirements to the computer time.

*Acknowledgements.* This work has been supported in part by Swiss National Science Foundation (SCOPES grant 128040).

## References

- [1] J.F. Navarro, C.S. Frenk, S.D.M. White, *The structure of cold dark matter halos*. ApJ, **462**, P.563-575 (1996).
- [2] J.F. Navarro, C.S. Frenk, S.D.M. White, *A universal density profile from hierarchical clustering*. ApJ, **490**, P.493-508 (1997).
- [3] V. Avila-Reese, P. Colín, O. Valenzuela, E. D’Onghia, C. Firmani, *Formation and Structure of Halos in a Warm Dark Matter Cosmology*. ApJ, **559**, Is.2, P.516-530 (2001).
- [4] P. Bode, J.P. Ostriker, N. Turok, *Halo Formation in Warm Dark Matter Models*. ApJ, **556**, Is.1, P.93-107 (2001).
- [5] T. Goerdt, B. Moore, J.I. Read, J. Stadel, M. Zemp, *Does the Fornax dwarf spheroidal have a central cusp or core?* MNRAS, **368**, Is.3, P.1073-1077 (2006).
- [6] A. Schneider, R.E. Smith, A.V. Macciò, B. Moore, *Non-linear evolution of cosmological structures in warm dark matter models*. MNRAS, **424**, Is.1, pp. 684-698 (2012).
- [7] A. Boyarsky, J. Lesgourgues, O. Ruchayskiy, M. Viel, *Lyman- $\alpha$  constraints on warm and on warm-plus-cold dark matter models*. JCAP, **05** id. 012 (2009).
- [8] V. Springel, *The cosmological simulation code GADGET-2*. MNRAS **364**, P.1105-1134 (2005).
- [9] J. Brandbyge, S. Hannestad, T. Hangbolle, B. Thomsen, *The Effect of Thermal Neutrino Motion on the Non-linear Cosmological Matter Power Spectrum*. JCAP, **08**, id. 020, 16 pp. (2008).
- [10] J. Brandbyge and S. Hannestad, *Grid Based Linear Neutrino Perturbations in Cosmological N-body Simulations*. JCAP, **05**, id.002 (2009).
- [11] F. Bernardeau, S. Colombi, E. Gaztañaga, R. Scoccimarro. *Large-scale structure of the Universe and cosmological perturbation theory*. Phys. Rep. **367**, Is.1-3, P.1-248 (2002)



- [12] A. Taruya and T. Hiramatsu. *A Closure Theory for Nonlinear Evolution of Cosmological Power Spectra*. Astrophys. J. **674**, Issue 2, pp. 617-635 (2008).
- [13] Y. Y. Y. Wong, *Higher order corrections to the large scale matter power spectrum in the presence of massive neutrinos*. JCAP, **10**, id. 035, 24 pp. (2008).
- [14] M. Pietroni. *Flowing with time: a new approach to non-linear cosmological perturbations*. JCAP, **10**, id.19 (2008), 19 pp.
- [15] J.Lesgourgues, S.Matarrese, M.Pietroni, A.Riotto. *Non-linear power spectrum including massive neutrinos: the time-RG flow approach*. JCAP, **06**, id.017 (2009).
- [16] J. Carlson, M. White, N. Padmanabhan. *A critical look at cosmological perturbation theory techniques*. Phys.Rev. **D80**, 043531 (2009)

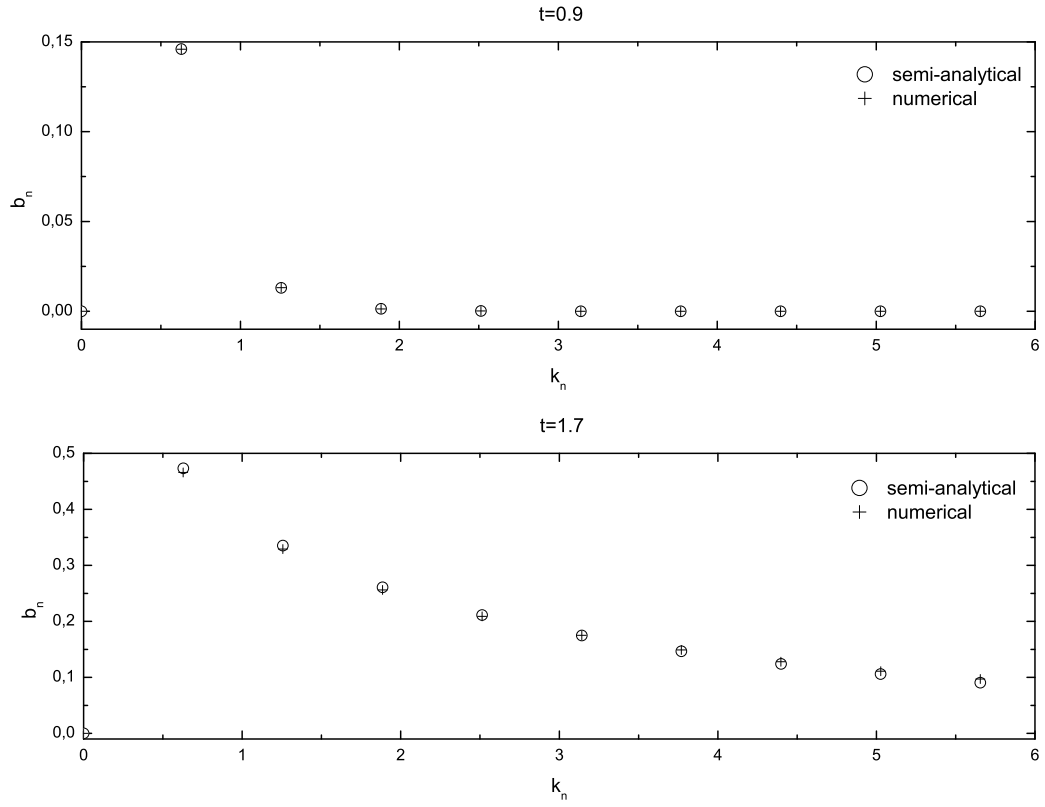


Figure 1: Coefficients  $b_n(t)$  for  $t = 0.9$  (top) and  $t = 1.7$  (bottom) determined by semi-analytical and numerical methods with the only nonzero initial values  $b_{\pm 1}(0) = 0.1$ .

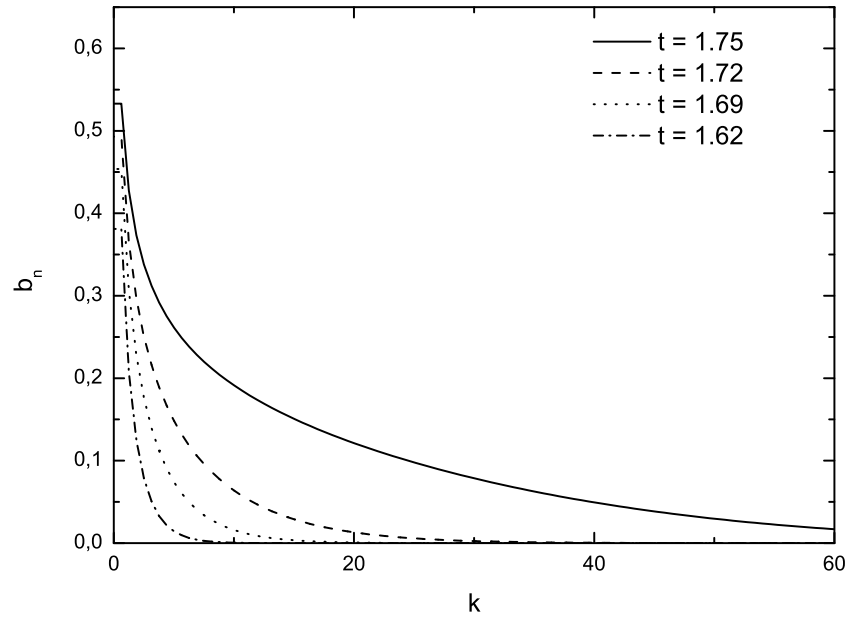


Figure 2: The evolution of perturbations over  $k$  for the initial conditions as described on Fig. 1.

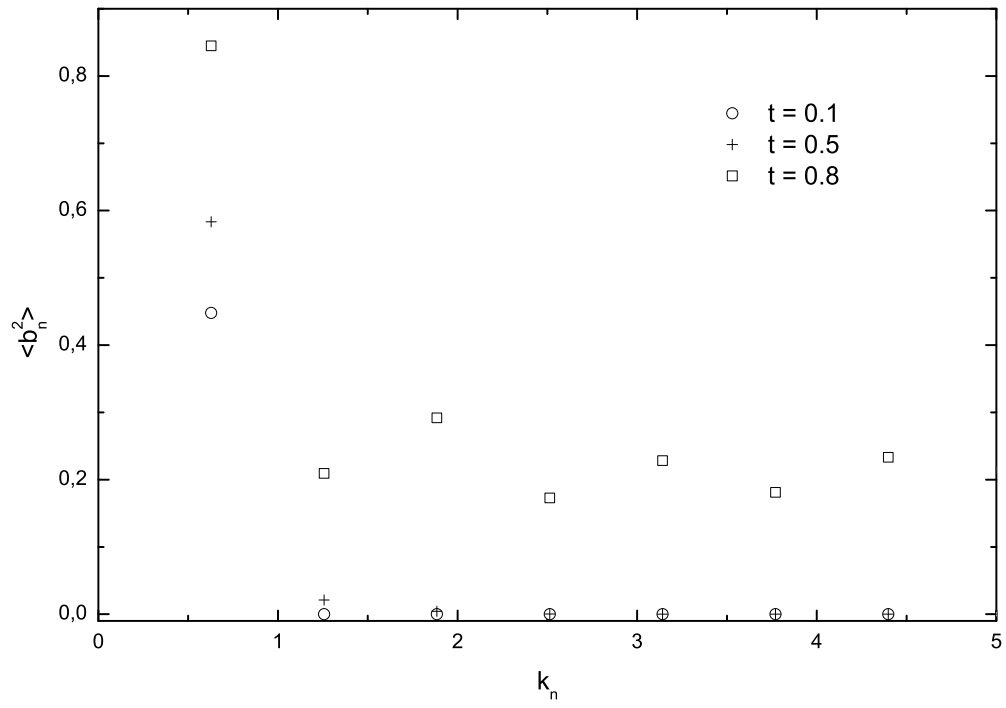


Figure 3: Power spectrum  $\langle b_n^2 \rangle$  calculated for 100 realizations of randomly generated independent initial conditions.